Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis -I

Mid-term Examination — Date : 11 October 2022

Maximum marks: 100 — Time: 3 hours

Instructor: B V Rajarama Bhat

(1) Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions and let $h : A \to C$ be defined by $h = g \circ f$. (i) Show that if h is injective then f is injective, but g may not be injective. (ii) Give an example where f is injective and g is surjective but h is neither injective nor surjective. [15]

(2) Let $\{b_n\}_{n\geq 1}$ be a sequence defined by $b_1 = 2, b_2 = 5$ and $b_{n+1} = 5b_n - 6b_{n-1}$ for $n \geq 2$. Show that $b_n = 2^{n-1} + 3^{n-1}$ for all $n \in \mathbb{N}$. [15]

(3) Let M be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. Show that M is uncountable. [15]

(4) Show that the set $A := \{3^n : n \in \mathbb{N}\}$ is not bounded above. [15]

(5) (i) Show that for a real number c if $c \in [-1, 1]$,

$$|c^n| \leq |c|, \ \forall n \in \mathbb{N}.$$

(ii) If $\{c_n\}_{n\geq 1}$ is a sequence of real numbers converging to 0, show that $\{c_n^n\}_{n\geq 1}$ converges to 0. [15]

(6) Show that there exists unique positive real number s such that $s^2 = 5$. Denote this real number by $\sqrt{5}$. Show that $\sqrt{5}$ is irrational. [15]

(7) Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers converging to 0. Assume that $a_n \geq a_{n+1}$ for every natural number n. Define a sequence of real numbers $\{s_n\}_{n\geq 1}$ by

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n.$$

That is, $s_1 = a_1$ and $s_{n+1} = s_n + (-1)^n a_{n+1}, \ \forall n \geq 1$. Show that $\{s_n\}_{n\geq 1}$ is convergent. [15]